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Ising Models, Julia Sets, and Similarity of the Maximal Entropy Measures

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We study the phase transition of Ising models on diamondlike hierarchical lattices. Following an idea of Lee and Yang, one can make an analytic continuation of free energy of this model to the complex temperature plane. It is known that the Migdal–Kadanoff renormalization group of this model is a rational endomorphism (denoted by f) of $\hat{\mathbb{C}}$ and that the singularities of the free energy lie on the Julia set $J(f)$. The aim of this paper is to prove that the free energy can be represented as the logarithmic potential of the maximal entropy measure on $J(f)$. Moreover, using this representation, we can show a close relationship between the critical exponent and local similarity of this measure.

KEY WORDS: Ising models; diamondlike hierarchical lattices; renormalization groups; Julia sets; maximal entropy measures; fractal structure.

1. INTRODUCTION

In this article we study the phase transition of the free energy $\mathscr{F}$ of Ising models. The phase transition is formulated as the nonanalyticity of physical quantities such as $\mathscr{F}$ as a function of some thermodynamic parameters such as temperature. So it is important to know where $\mathscr{F}$ is analytic and how $\mathscr{F}$ behaves near nonanalytic (critical) points, because it determines the type of the phase transition.

Here we study the phase transition of an exactly solvable model, the Ising model on diamondlike hierarchical lattices. The diamondlike hierarchical lattices construct a sequence of graphs $\{\Gamma_n\}$ defined as follows. For a fixed integer b greater than one, we define two lattices $\Gamma_G = \{B_G, V_G\}$ (generator) and $\Gamma_0 = \{B_0, V_0\}$ (initial lattice), where B_* denotes the set of all bonds of Γ_*, and V_* denotes the set of all vertices of Γ_*. Γ_0

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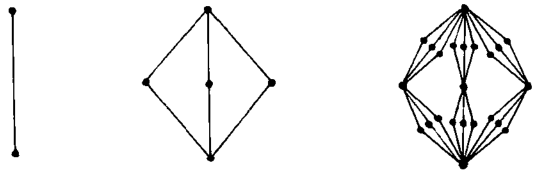

Γ_0 Γ_1 Γ_2

Fig. 1. Diamondlike hierarchical lattices for $b = 3$.

consists of two vertices and one bond connecting them. To obtain Γ_G, we insert b inner vertices between the two outer ones such that each bond connects an inner vertex and an outer one. When Γ_n is constructed, Γ_{n+1} is obtained by replacing each element of B_n by Γ_G. We call $\Gamma_n = \{B_n, V_n\}$ diamondlike hierarchical lattices.[4,5] (See Fig. 1 for the case $b = 3$.)

Bleher and Zalis[4] showed that the free energy on these lattices is discribed as

$$\mathscr{F}(T) = -\frac{J}{2} - \frac{T}{2} \sum_{n=0}^{\infty} \frac{1}{(2b)^n} \log(1 + t_n^b)$$

where

$$t_0 = \exp\left(\frac{2J}{bT}\right)$$

$$t_{n+1} = f(t_n)$$

$$f(t) = \frac{4t^b}{(1 + t^b)^2}$$

The map f is called the Migdal–Kadanoff renormalization group transformation, which is a strong tool to investigate the asymptotic behavior of the free energy near the critical temperature. The advantage of this model is that the renormalization group transformation can be expressed explicitly and, moreover, it turns out to be a rational map.

In the theory of statistical mechanics the following quantity which discribes the asymptotic behavior of the free energy is useful for a characterization of the phase transition. The critical exponent in the low-temperature region of order $l \in \mathbf{N}$ is defined by

$$\alpha_{t_c}^{(l)} \equiv \lim_{t \nearrow t_c} \frac{\log |\mathscr{F}^{(l)}(t)|}{-\log |t - t_c|}$$

where t_c is the critical temperature and $\mathscr{F}^{(l)}$ is the lth derivative of $\mathscr{F}$.

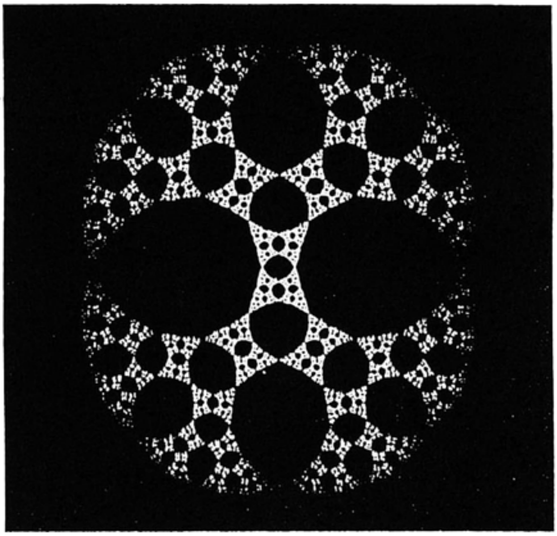

Fig. 2. Julia set of f for $b=2$.

Lee and Yang[1] suggested an approach for the study of the singularities of the free energy. As is seen in the definition of free energy, the singularities of the free energy appear at zeros of the partition functions. But it is shown that each partition function is essentially a polynomial of t with positive coefficients. So the singularities never appear on the temperature interval $[0, 1]$. Lee and Yang proposed to extend the temperature t to the complex plane. They claimed that, letting $n \to \infty$, the zeros in the complex plane approach some points of $[0, 1]$, which would represent the phase transition points. Following this idea, we consider $\mathscr{F}$ on $\mathbf{C}$ and f as a dynamical system on $\hat{\mathbf{C}}$.

The theory of complex dynamical systems has been developed in recent years. One of the main objects of this theory is to study the invariant set called the Julia set $J(f)$, where the dynamics of f is "chaotic." Figure 2 shows the Julia set for $b=2$. For the study of dynamics of f on $J(f)$, Brolin[9] introduced (for the polynomial case) a natural masure μ called the maximal entropy measure, the support of which coincides with $J(f)$.

The purpose of this paper is to show a relationship between the two theories, statistical and complex dynamical systems. Historically, Derrida *et al.*[6] showed such a relationship for the first time. In ref. 6 it was shown that all the singularities of $\mathscr{F}$ lie on the Julia set of the renormalization group transformation. Our main results are more concerned with the quantitative relationship between them.

Theorem A. The free energy can be represented as the logarithmic potential of the maximal entropy measure μ,

$$F(t)=b\int_{J(f)}\log(t-z)\,d\mu(z)+C$$

Using this representation, we can show the following.

Theorem B. Let l be so large that $(f'(t_c))^l>2b$; then we have

$$\alpha^{(l)}=l-\frac{\log 2b}{\log f'(t_c)}$$

Moreover, $\log 2b/\log f'(t_c)$ represents the local similarity near t_c of the measure μ on $J(f)$.

Thus, Theorem B says that the phase transition reflects the fractal structure of $J(f)$.

2. DYNAMICS OF THE RENORMALIZATION GROUP

By the change of variable $t=\exp(-2J/bT)$, the temperature interval $[0,\infty]$ is mapped onto $[0,1]$. So, first we consider f as a dynamical system on $[0,1]$. Then we can check that $t=0$, 1 are superattractive fixed points of f and there exists a unique repelling fixed point t_c in $(0,1)$.

From now on, we consider

$$F(t)=\sum_{n=0}^{\infty}\frac{1}{(2b)^n}g\circ f^n(t), \qquad g(s)=\log(1+s^b) \tag{2.1}$$

instead of $\mathscr{F}$, where f^n is the n-fold iterate of f.

As explained in the introduction, we consider $\mathscr{F}$ as a function on $\mathbf{C}$ and f as a dynamical system on $\hat{\mathbf{C}}$. Let Ω_0 be the immediate attractive basin of 0. One can show that F is analytic on Ω_0 and, moreover, $\partial\Omega_0$ forms a natural boundary of F.

From the physical point of view we are interested in the behavior of $F^{(l)}$ (the lth derivative of F) when t approaches $\partial\Omega_0$. Bleher and Lyubich[5] proved the following.

Theorem (Bleher–Lyubich[5]). Let b be greater than 2. Then:

1. $F^{(2)}(t)$ is not continuous up to $\partial\Omega_0$.
2. For almost all geodesics from the origin to a point $\tau \in \partial\Omega_0$ with respect to the harmonic masure, the critical exponents along the geodesics become

$$\alpha_\tau^{(2)} = 1 - \frac{\log 2}{\log b}$$

Remark that we cannot check whether this result is valid for $\alpha_{t_c}^{(l)}$ or not. To calculate $\alpha_{t_c}^{(l)}$, we introduce a "natural" measure μ on $J(f)$. This measure was introduced by Brolin[9] for the polynomial case. First it is not difficult to see that all the critical points of f are eventually mapped to the superstable fixed points 0, 1. So the dynamics f on $J(f)$ is expanding. By the Bowen–Ruelle–Sinai theorem, there exists a unique equilibrium state μ satisfying the variational principle for potential $\rho \equiv 0$,

$$P_f(0) = h_\mu(f) = \sup_{\nu \in M(f)} h_\nu(f)$$

where $M(f)$ denotes all f-invariant probability measures on $J(f)$. Here $P_f(0)$ is just the topological entropy (in this case, it is $\log 2b$), so μ is called the maximal entropy measure. This measure is uniquely characterized by the following fact.

Proposition (Freire–Lopes–Mañé,[10] Mañé,[11] Lyubich[12]). For any Borel set A where $f\rceil_A$ is injective, we have

$$\mu(f(A)) = (2b) \cdot \mu(A) \tag{2.2}$$

Conversely, the maximal entropy measure is the unique f-invariant probability measure satisfying the above equation.

3. REPRESENTATION OF THE FREE ENERGY F

In this section we prove that the free energy F is represented as the logarithmic potential of μ. First we need the following.

Lemma. For $z \in J(f)$ and $t \in \mathbf{C} \backslash J(f)$, we have

$$\frac{f(t) - z}{-z} = \frac{\prod_{i=1}^{2b} (t - z_i)}{(1 + t^b)^2}$$

where z_i $(1 \leqslant i \leqslant 2b)$ are the inverse images of z counting multiplicity.

Proof. For $z \in J(f)$, consider the following polynomial of t of degree $2b$:

$$p(t) = (1 + t^b)^2 - \frac{1}{z} \cdot 4t^b$$

Then, we can easily see that

$$p(t) = 0 \Leftrightarrow z = \frac{4t^b}{(1 + t^b)^2} = f(t) \Leftrightarrow t = z_i$$

Thus, one gets

$$p(t) = D \prod_{i=1}^{2b} (t - z_i)$$

Comparing the coefficient of t^{2b}, we get $D = 1$. This proves the lemma. ∎

Using this lemma, we can show the following integral representation of the free energy.

Proof of Theorem A. Let $F_1(t)$ be the right-hand side of (2.1), and let $F_2(t)$ be the integral representation of $F(t)$. Remark that $F_1(0) = 0$ and choose a constant C so that $F_2(t) = 0$. Consider a functional equation

$$E(t) = \frac{1}{2b} E \circ f(t) + g(t) \tag{3.1}$$

It is easy to see that both F_1 and F_2 satisfy (3.1). In fact, using the previous lemma and (2.2), one gets

$$\begin{aligned}
&\frac{1}{2b} F_2 \circ f(t) \\
&\quad = \frac{1}{2b} b \int_{J(f)} \log[f(t) - z] \, d\mu(z) - \frac{1}{2b} b \int_{J(f)} \log(-z) \, d\mu(z) \\
&\quad = \frac{1}{2} \int_{J(f)} \log \frac{\prod_{i=1}^{2b} (t - z_i)}{(1 + t^b)^2} \, d\mu(t) + i\pi n_1(t) \\
&\quad = \frac{1}{2} \sum_{i=1}^{2b} \int_{J(f)} \log(t - z_i) \, d\mu(z) - \frac{1}{2} \log(1 + t^b)^2 \int_{J(f)} d\mu(z) + i\pi n_2(t) \\
&\quad = \frac{1}{2} \int_{J(f)} \log(t - z) \, d\mu(f(z)) - \log(1 + t^b) + i\pi n_3(t) \\
&\quad = \frac{1}{2} \cdot 2b \int_{J(f)} \log(t - z) \, d\mu(z) - \log(1 + t^b) + i\pi n_3(t) \\
&\quad = F_2(t) - g(t) + i\pi n_3(t)
\end{aligned}$$

where $n_i(t) \in \mathbf{N}$ appear by choosing branches of logarithms. Here

$$i\pi n_3(t) = \frac{1}{2b} F_2 \circ f(t) - F_2(t) + g(t)$$

is continuous, and $n_3(t)$ equals a constant. Letting $t = 0$, we get $n_3(t) = 0$.

So we must claim the uniqueness of the continuous solution $E(t)$ of (3.1) satisfying $E(0) = 0$. Let $G(t) \equiv F_2(t) - F_1(t)$. Then G must satisfy

$$\frac{1}{2b} G \circ f(t) - G(t) = 0, \qquad G(0) = 0 \tag{3.2}$$

Assume that $G(t_0) \neq 0$ for some $t_0 \in \Omega_0$. Then, using (3.2) inductively, we have

$$G \circ f^n(t_0) = (2b)^n \cdot G(t_0)$$

Because $t_0 \in \Omega_0$, $f^n(t_0)$ goes to 0 as n increases. Thus by the continuity of G, $G \circ f^n(t_0) \to 0$. But $(2b)^n \cdot G(t_0) \to \infty$; this is a contradiction. ∎

In fact, the same statement holds for any t in $\mathbf{C} \backslash J(f)$ because in the expanding case each component of $\mathbf{C} \backslash J(f)$ is the preimage of Ω_0 or Ω_1 (immediate attractive basin of 1).

4. THE CRITICAL EXPONENT AT t_c

In this section we establish a relationship between the real critical exponent and the maximal entropy measure, using the representation in Theorem A. A similar equation was already conjectured in ref. 8.

Proof of Theorem B. Let $r > 0$ be small enough and take a disk $B_r(t_c) = \{z \in \mathbf{C} \mid |z - t_c| \leqslant r\}$. Let $J_r \equiv J(f) \cap B_r(t_c)$, and take an arbitrary $p_0 \in (0, t) \cap B_r(t_c)$. Define a sequence $p_n \in (p_{n-1}, t_c)$ so that $p_{n-1} = f(p_n)$. Let f^{-1} be the inverse branch on J_r which fixes t_c. Consider the ratio

$$\begin{aligned} \frac{F^{(l)}(p_{n+1})}{F^{(l)}(p_n)} = &\left\{ c_l \int_{f^{-1}(J_r)} \frac{d\mu(t)}{(t - p_{n+1})^l} + c_l \int_{J(f) \backslash f^{-1}(J_r)} \frac{d\mu(t)}{(t - p_{n+1})^l} \right\} \\ &\times \left\{ c_l \int_{J_r} \frac{d\mu(t)}{(t - p_n)^l} + c_l \int_{J(f) \backslash J_r} \frac{d\mu(t)}{(t - p_n)^l} \right\}^{-1} \end{aligned} \tag{4.1}$$

Let $t' = f(t)$. Then, by the proposition in Section 2, one gets

$$\int_{f^{-1}(J_r)} \frac{d\mu(t)}{(t - p_{n+1})^l} = \frac{1}{2b} \int_{J_r} \frac{d\mu(t')}{[f^{-1}(t') - f^{-1}(p_n)]^l} \tag{4.2}$$

From the definition of the derivative we see that

$$\frac{1}{[f^{-1}(t')-f^{-1}(p_n)]^l}-\left(\frac{f'(t_c)}{t'-p_n}\right)^l=\frac{a_n+d_r}{t'-p_n}\left(\frac{f'(t_c)}{t'-p_n}\right)^{l-1}+\cdots$$
$$=(a_n+d_r)\frac{k_l}{(t'-p_n)^l}$$

where a_n goes to zero as n tends to infinity and d_r goes to zero as r tends to zero. Thus one gets

$$(4.2)=\left[\frac{f'(t_c)^l}{2b}+k_l(a_n+d_r)\right]\int_{J_r}\frac{d\mu(t')}{(t'-p_n)^l}$$

For a fixed $r>0$, the first terms in the numerator and the denominator of (4.1) go to infinity and the second ones are bounded as $n\to\infty$. So the ratio (4.1) approaches

$$\frac{[f'(t_c)]^l}{2b}+d_r\cdot k_l$$

as n tends to infinity. But, as r is arbitrarily chosen, we have

$$\log F^{(l)}(p_{n+1})=\sum_{k=1}^{n}\log\left(\frac{[f'(t_c)]^l}{2b}+a'_k\right)+\log F^{(l)}(p_0) \tag{4.3}$$

where $a'_n\to 0$.

On the other hand, one can easily get

$$\log(t_c-p_{n+1})=\log(t_c-p_0)-\sum_{k=1}^{n}\log[f'(t_c)+b_n] \tag{4.4}$$

where $b_n\to 0$ as $n\to\infty$. From Eqs. (4.3) and (4.4), ignoring constant terms, one has

$$\begin{aligned}
\lim_{n\to\infty}&\left|\frac{\log F^{(l)}(p_{n+1})}{-\log(t_c-p_{n+1})}-\frac{\log\{[f'(t_c)]^l/2b\}}{\log f'(t_c)}\right| \\
&=\lim_{n\to\infty}\left|\left\{\log f'(t_c)\cdot\sum_{k=1}^{n}\log\left(\frac{[f'(t_c)]^l}{2b}+a'_k\right)\right.\right. \\
&\quad\left.-\log\frac{[f'(t_c)]^l}{2b}\cdot\sum_{k=1}^{n}\log[f'(t_c)+b_k]\right\} \\
&\quad\times\left.\left\{\log f'(t_c)\cdot\sum_{k=1}^{n}\log[f'(t_c)+b_n]\right\}^{-1}\right|
\end{aligned}$$

$$= \lim_{n\to\infty} \left| \frac{\sum_{k=1}^{n} \{a'_k \log f'(t_c) - b_k \log[f'(t_c)^l/2b]\}}{n \cdot \log f'(t_c)} \right|$$

$$= 0$$

Finally we can get

$$\lim_{n\to\infty} \frac{\log F^{(l)}(p_n)}{-\log(t_c - p_n)} = l - \frac{\log 2b}{\log f'(t_c)}$$

Using the boundedness of $F^{(l)}(t)$ on $[p_0, p_1]$, it is not difficult to deduce our statement from this. ■

Remark 1. In the same way, we can show that the phase transition does not occur in the case of $[f'(t_c)]^l < 2b$. We do not know what happens when $[f'(t_c)]^l$ equals $2b$.

5. LOCAL SIMILARITY OF μ

What does $\log 2b/\log f'(t_c)$ in Theorem B mean? First consider, for example, the Sierpiński gasket (Fig. 3). When we enlarge the size of the Sierpiński gasket twice, the "area" (rigorously speaking, the Hausdorff measure) increases three times. Thus, the similarity dimension of the Sierpiński gasket equals $\log 3/\log 2$. This is the fundamental idea of the similarity dimension.

In our case, if we linearize f near t_c, Eq. (5.1) becomes

$$\mu(L(V)) \approx (2b) \cdot \mu(V)$$

where $L(t) \equiv f'(t_c)(t - t_c) + t_c$ is the linearization of f, and V is a neighborhood of t_c. See Fig. 4, where we can see the fractal structure of $J(f)$ near t_c, and convince ourselves that the above equation is quite precise.

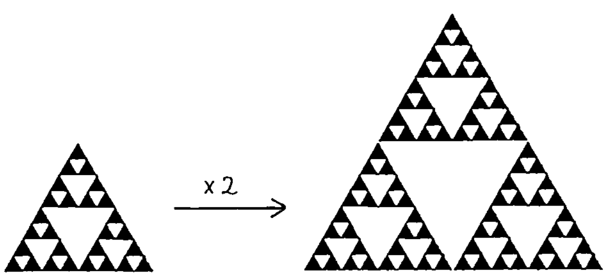

Fig. 3. Sierpiński gasket.

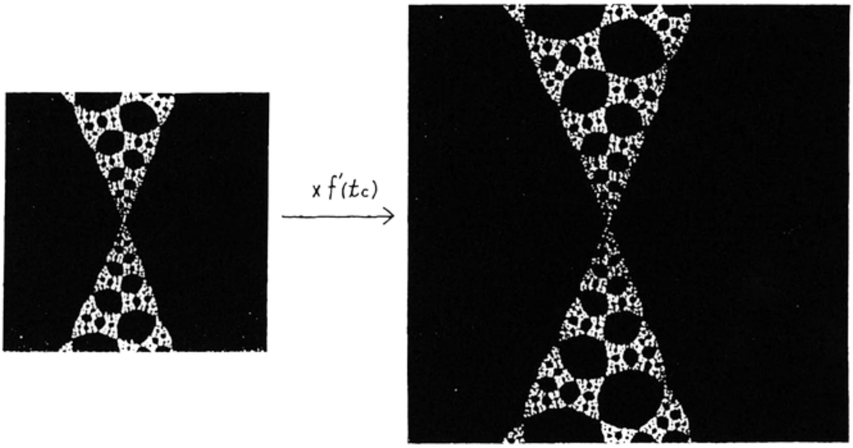

Fig. 4. Enlargement of $J(f)$ near t_c.

This situation is just the same as the case of the Sierpiński gasket. That is, when we enlarge the size of V $f'(t_c)$ times, the measure $\mu(V)$ becomes about $2b$ times greater. So $\log 2b/\log f'(t_c)$ is supposed to represent the similarity of μ near t_c. Thus, Theorem B states that the critical exponent reflects the local similarity of the maximal entropy measure.

Remark 2. The following remark is due to Dr. H. Kokubu on a connection between the results of Bleher and Lyubich and the author. If we rewrite the result of Bleher and Lyubich in our fashion, we get

$$\alpha_\tau^{(2)} = 2 - \frac{\log 2b}{\log b}$$

So in both cases the critical exponent shows the following form:

$$l - \frac{\text{topological entropy of } f}{\text{Lyapunov exponent at } \tau}$$

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